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A systematic method for the solution of some nonlinear evolution equations: I. The Burgers equations

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Abstract. The eigenfunction method introduced by Van Kampen to solve a Fokker–Planck equation is extended and applied to a related partial differential equation. From this analysis we are able to obtain the solution of the Burgers and the damped Burgers equation in a systematic way.

1. Introduction

In the last decade, different methods have become available for solving nonlinear wave equations. Among these are the powerful inverse scattering technique (IST) (Gardner *et al* 1974) and the Hirota (1976) method which is based on the theory of Padé approximations. These methods give very good results, and even complicated nonlinear equations can be solved using them.

Nevertheless, we intend to introduce another method, because we feel that a more systematic approach is needed. For example, the Schrödinger equation which appears in the IST method for the solution of the KDV equation seems somewhat 'pulled from the air' (Scott *et al* 1973). Furthermore, Hirota's method is based on a trial solution in the form of the ratio of two functions, which was a useful hypothesis. The reason that one has to make some kind of a guess to start with is clear: we are dealing with nonlinear equations and every effort to classify them fails as yet.

In this article we propose an attempt at a better starting point for some evolution equations.

We shall deal now with the Burgers equation; the Korteweg–de Vries equation will be studied in a subsequent paper.

2. The eigenfunction method of Van Kampen

Recently, Van Kampen (1977) found an explicit solution of the one-dimensional Fokker–Planck equation

$$\frac{\partial P}{\partial t}=\frac{\mathrm{d}^2U}{\mathrm{d}x^2}P+\frac{\mathrm{d}U}{\mathrm{d}x}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 P}{\partial x^2}\qquad(\nu=\text{constant})\tag{1}$$

which originates from a stochastic description of a bistable system. The quantity $P(x, t)$

in this context is associated with a probability function, while $U(x)$ represents a (not explicitly known) potential function with two minima.

The scheme of this method can be summarised as follows. Firstly, a time factor $e^{-\lambda t}$ is split off. The remaining eigenvalue equation then reads

$$\nu\frac{d^2P}{dx^2}+\frac{dU}{dx}\frac{dP}{dx}+\left(\frac{d^2U}{dx^2}+\lambda\right)P=0, \tag{2}$$

where P is now only a function of x.

Secondly, if one substitutes

$$P(x)=\exp[-U(x)/2\nu]\phi(x), \tag{3}$$

equation (2) can be reduced to the canonical form

$$\frac{d^2\phi}{dx^2}+\left[-\frac{1}{4\nu^2}\left(\frac{dU}{dx}\right)^2+\frac{1}{2\nu}\frac{d^2U}{dx^2}+\frac{\lambda}{\nu}\right]\phi=0. \tag{4}$$

Then one defines a function $V(x)$

$$V(x)=\frac{1}{4\nu^2}\left(\frac{dU}{dx}\right)^2-\frac{1}{2\nu}\frac{d^2U}{dx^2}+C \qquad (C=\text{constant}) \tag{5}$$

such that equation (4) reduces to a Schrödinger equation:

$$d^2\phi/dx^2+[E-V(x)]\phi=0. \tag{6}$$

Further, the RHS of equation (5) resembles a Riccati equation which can be transformed by the usual substitution

$$U=-2\nu\ln Z \qquad \text{or} \qquad Z=\exp(-U/2\nu). \tag{7}$$

The result reads

$$d^2Z/dx^2+[C-V(x)]Z=0, \tag{8}$$

which again represents a Schrödinger equation with the same potential!

Finally, the quantity P we are looking for can then be written as a product of the eigenfunctions ϕ and Z.

The features of this method can be summarised as follows. If $U(x)$ is explicitly known, the problem is reduced to the Schrödinger equation (4). If only the form or an expression for $U(x)$ is known (as in Van Kampen's case), one can look for a positive eigenfunction Z in order to find a suitable form for $V(x)$ and hence $U(x)$.

This second feature will be very important for our following analysis.

3. Extension of the method

We intend to investigate the equation

$$\frac{\partial P}{\partial t}=\alpha\frac{\partial^2U}{\partial x^2}P+\beta\frac{\partial U}{\partial x}\frac{\partial P}{\partial x}+\nu\frac{\partial^2P}{\partial x^2}, \tag{9}$$

where α and β are parameters and U is an arbitrary function of x and t.

This equation is closely related to our starting differential equation (1). An extra freedom, however, is that $U(x, t)$ may be time dependent. Nevertheless we shall make use of the preceding analysis.

Due to the time-dependent character of U we cannot split off a time factor. Therefore we immediately substitute

$$P(x, t)=\exp[-\beta(U(x, t)/2\nu)]\phi(x, t). \tag{10}$$

Hence equation (9) is written as

$$\frac{\partial \phi}{\partial t}=\nu\frac{\partial^2\phi}{\partial x^2}+\frac{\beta\phi}{2\nu}\left[\frac{\partial U}{\partial t}-\nu\frac{(\beta-2\alpha)}{\beta}\frac{\partial^2 U}{\partial x^2}-\frac{\beta}{2}\left(\frac{\partial U}{\partial x}\right)^2\right]. \tag{11}$$

The Riccati-like expression on the RHS of this equation appears again, and consequently we are able to linearise it by the usual substitution

$$U(x, t)=2\nu a\ln Z \qquad \text{or} \qquad Z=\exp[U(x, t)/2\nu a] \tag{12}$$

with

$$a=(\beta-2\alpha)/\beta^2.$$

This choice for the parameter a enables us to cancel the nonlinear term $-(\beta/2)(\partial U/\partial x)^2$. Then we arrive at

$$\frac{\partial \phi}{\partial t}=\nu\frac{\partial^2\phi}{\partial x^2}+\frac{\beta a\phi}{Z}\left(\frac{\partial Z}{\partial t}-a\beta\nu\frac{\partial^2 Z}{\partial x^2}\right). \tag{13}$$

This equation can be readily solved by taking

$$\partial Z/\partial t=a\beta\nu\,\partial^2 Z/\partial x^2+CZ \tag{14a}$$

and

$$\partial\phi/\partial t=\nu\,\partial^2\phi/\partial x^2+\beta aC\phi, \tag{14b}$$

where C is an arbitrary function of x and t. Both the quantities ϕ and Z obey the same type of equation. For $C=0$ the familiar thermal conductivity equation is obtained.

We remark that the function $C(x, t)$ here plays the same role as the potential function $V(x)$ in § 2.

Special cases of this solution are the following.

(1) $\alpha=\beta=1$, $a=-1$.
We have

$$Z=\exp(-U/2\nu), \qquad P=\phi Z, \qquad \partial Z/\partial t=-\nu\,\partial^2 Z/\partial x^2+CZ$$

and

$$\partial\phi/\partial t=\nu\,\partial^2\phi/\partial x^2-C\phi. \tag{15}$$

The solution for P can thus be written as a product of two functions. This example just generalises van Kampen's treatment.

(2) $\alpha=0$, $\beta=1$, $a=1$.
Then

$$Z=\exp(U/2\nu) \qquad \text{and} \qquad P=\phi/Z$$

with

$$\partial Z/\partial t=\nu\,\partial^2 Z/\partial x^2+CZ$$

and

$$\partial\phi/\partial t = \nu\, \partial^2\phi/\partial x^2 + C\phi. \tag{16}$$

We now have to deal with the ratio of two functions which are related because they obey the same equation. Note that one has to exclude $\beta = 2\alpha$ because the expression (12) becomes meaningless.

In conclusion, a solution of equation (9) can be found if we take a function U which obeys the Riccati-like equation

$$\frac{\partial U}{\partial t} - \nu\alpha\beta\frac{\partial^2 U}{\partial x^2} - \frac{\beta}{2}\left(\frac{\partial U}{\partial x}\right)^2 = 2\nu\alpha C. \tag{17}$$

Then equation (11) can be separated into two equations and the corresponding solutions can be determined.

4. Burgers' equation

This well known equation, first introduced by Burgers (1948), serves as a useful mathematical and physical model in fluid mechanics. Moreover it represents the simplest nonlinear wave equation:

$$\frac{\partial P}{\partial t} = -P\frac{\partial P}{\partial x} + \nu\frac{\partial^2 P}{\partial x^2}. \tag{18}$$

The solution of this equation, proposed by Cole and Hopf, is remarkable (Karpman 1975). By means of the transformation

$$P = -2\nu\frac{\partial}{\partial x}\ln\psi = -2\nu\frac{\partial\psi}{\partial x}\Big/\psi, \tag{19}$$

they obtained for ψ a closed linear analytical form:

$$\partial\psi/\partial t = \nu\, \partial^2\psi/\partial x^2. \tag{20}$$

The key to that transformation seems to be a guess or just a coincidence.

Let us now examine this nonlinear equation by the present method. A connection between equation (9) and equation (18) is easily found if we postulate the relations

$$\partial U/\partial x = -P \tag{21a}$$

and

$$\alpha + \beta = 1. \tag{21b}$$

As mentioned before, we are able to use our solution method only for those functions U which obey equation (17). Such an equation is available. Indeed, from equation (21a) we observe that $-\partial U/\partial x$ has to satisfy Burgers' equation. This gives

$$\frac{\partial}{\partial x}\left[\frac{\partial U}{\partial t} - \frac{1}{2}\left(\frac{\partial U}{\partial x}\right)^2 - \nu\frac{\partial^2 U}{\partial x^2}\right] = 0 \tag{22a}$$

or

$$\frac{\partial U}{\partial t} - \frac{1}{2}\left(\frac{\partial U}{\partial x}\right)^2 - \nu\frac{\partial^2 U}{\partial x^2} = c'(t). \tag{22b}$$

Equation (17) is similar to equation (22*b*) if we choose

$$\alpha = 0, \qquad \beta = 1 \qquad (\text{hence } a = 1) \tag{23a}$$

and

$$C = c'(t)/2\nu = c(t) \tag{23b}$$

except for the fact that now C may be only time-dependent.

Our problem now corresponds to the second example we treated in the previous section. In that case (see equations (16)) we had

$$P = \phi/Z, \tag{24}$$

and because also

$$P = -\frac{\partial U}{\partial x} = -2\nu\frac{1}{Z}\frac{\partial Z}{\partial x}, \tag{25}$$

we arrive immediately at

$$\phi = -2\nu\, \partial Z/\partial x. \tag{26}$$

In view of the equations (16) and the fact that we deal with just one unknown variable, this relationship between ϕ and Z had to exist. Hence, equation (25) is a solution of Burgers' equation provided that Z satisfies the linear differential equation

$$\partial Z/\partial t = \nu\, \partial^2 Z/\partial x^2 + c(t)Z. \tag{27}$$

Note that equation (24) corresponds to the starting hypothesis of Hirota (1976) and that equation (25) is nothing but the Hopf–Cole transformation (see equation (19)).

We notice further that equation (27) is more general than originally stated. Hirota (1976) has already pointed out that a term $\lambda\psi$ (λ an arbitrary constant) has to be added to the equation of thermal conductivity (20).

5. The damped Burgers equation

The damped Burgers equation is just a simple extension of equation (18):

$$\frac{\partial P}{\partial t} + \gamma P = -P\frac{\partial P}{\partial x} + \nu\frac{\partial^2 P}{\partial x^2}. \tag{28}$$

The constant $\gamma(>0)$ represents a damping rate.

According to the previous analysis, equation (28) is transformed into the more suitable form

$$\frac{\partial P}{\partial t} + \gamma P = \frac{\partial U}{\partial x}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 P}{\partial x^2} \tag{29}$$

and

$$\partial U/\partial x = -P. \tag{30}$$

The usual substitution $P = \exp(-U/2\nu)\phi$ leads us to

$$\frac{\partial \phi}{\partial t} + \gamma\phi = \nu\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{2\nu}\phi\left[\frac{\partial U}{\partial t} - \frac{1}{2}\left(\frac{\partial U}{\partial x}\right)^2 - \nu\frac{\partial^2 U}{\partial x^2}\right]. \tag{31}$$

Substitution of the condition (30), i.e. $P=-\partial U/\partial x$, into equation (28) gives

$$\frac{\partial U}{\partial t}+\gamma U-\frac{1}{2}\left(\frac{\partial U}{\partial x}\right)^2-\nu\frac{\partial^2 U}{\partial x^2}=C'(t), \tag{32}$$

where $C'(t)$ is an arbitrary function of time. Hence equation (31) becomes

$$\frac{\partial \phi}{\partial t}+\gamma\phi=\nu\frac{\partial^2\phi}{\partial x^2}+\frac{C'}{2}\phi-\frac{1}{2\nu}\gamma U\phi. \tag{33}$$

Referring to the previous sections, we define

$$U=2\nu\ln Z. \tag{34}$$

Consequently the relation

$$\phi=-2\nu\,\partial Z/\partial x \tag{35}$$

is found again.

With the aid of these relations a closed expression for the function Z is discovered. Indeed, equation (33) is written as

$$\frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial t}+\gamma Z-\frac{C'(t)}{2\nu}Z\right)-\nu\frac{\partial}{\partial x}\frac{\partial^2 Z}{\partial x^2}=-\gamma\ln Z\frac{\partial Z}{\partial x} \tag{36}$$

or

$$\frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial t}+\gamma Z\ln Z+C_0(t)Z-\frac{\partial^2 Z}{\partial x^2}\right)=0, \tag{37}$$

with $C_0(t)=-C'(t)/2\nu$.

The final equation for Z is

$$\partial Z/\partial t+\gamma Z\ln Z=\nu\,\partial^2 Z/\partial x^2-C_0(t)Z, \tag{38}$$

if the expression between brackets of equation (37) vanishes as $|x|\to\infty$. Thus with the aid of

$$P=-(2\nu/Z)\,\partial Z/\partial x \tag{39}$$

the damped Burgers equation (28) is reduced to equation (38). Apart from the last term, equation (38) is similar to the solution suggested by Leibovich and Seebass (1974).

6. Conclusion

Firstly, we have extended the eigenfunction method of Van Kampen to a more general case. Then we were able to apply this analysis to the Burgers and damped Burgers equations in order to obtain the respective solutions in a systematic way.

These solutions are more general than those previously obtained. The ansatz used by Hirota appears as a logical consequence of our theory, and moreover we rediscover the Cole–Hopf transformation.

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